

On index of certain nilpotent Lie algebras

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Preliminary information and statements of theorems

The coadjoint orbits play an important role in representation theory, symplectic geometry and mathematical physics. According to the orbit method of A.A.Kirillov [1, 2] for nilpotent Lie groups there exists one to one correspondence between the coadjoint orbits and the irreducible unitary representations in Hilbert spaces. This gives a possibility to solve the problems of representation theory and harmonic analysis in geometrical terms of the orbit space. However the problem of classification of all coadjoint orbits for specific Lie groups (such as the group of unitriangular matrices) is an open problem up to today that is far from its solution. In the origin paper on the orbit method [2] the description of algebra of invariants and classification of orbits of maximal dimension was obtained.

In our paper we study the coadjoint representation for nilpotent Lie algebras that are factors of the unitriangular Lie algebra by ideals generated by subsets of root vectors. By this algebra \mathcal{L} we construct the diagram $\mathcal{D}_{\mathcal{L}}$ using the formal rule of placing of symbols in the table.

Applying this diagram \mathcal{D} one can easily calculate the index of concerned Lie algebra (see theorem 2(3)). Recall that the index of a Lie algebra is a minimal dimension of stabilizer of a linear form on this Lie algebra. For a nilpotent Lie algebra the field of invariants of the coadjoint representation is a pure transcendental extension of the main field of degree that is equal to the index of this algebra [5]. Respectively, by the diagram one can easily calculate the maximal dimension of coadjoint orbits (see theorem 2(2)). We also introduce the method of construction of system of generators in the field of invariants of the coadjoint representation (see theorem 1). The main results of the paper is stated in theorems 1 and 2 in the sequel of this section. .

Note that the author used the diagram in the previous papers for a classification of all coadjoint orbits for $n \leq 7$ [3], and also for a description of specific orbits

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for an arbitrary n (subregular orbits [3]; orbits associated with involutions [4]).

Let $N = \text{UT}(n, K)$ be a group of lower triangular matrices of size $n \times n$ with units on the diagonal and with entries in the field K of zero characteristic. The Lie algebra $\mathfrak{n} = \mathfrak{ut}(n, K)$ of this group consists of all lower triangular matrices of size $n \times n$ with zeros on the diagonal. The group N acts on the conjugate space \mathfrak{n}^* by the formula $\text{Ad}_g^* f(x) = f(\text{Ad}_g^{-1} x)$. This representation is called the coadjoint representation.

We identify the symmetric algebra $S(\mathfrak{n})$ with the algebra of regular functions $K[\mathfrak{n}^*]$ on the conjugate space \mathfrak{n}^* . We also identify \mathfrak{n}^* with the subspace of upper triangular matrices with zeros on the diagonal. The pairing of \mathfrak{n} and \mathfrak{n}^* is realized by the Killing form $(a, b) = \text{Tr}(ab)$, where $a \in \mathfrak{n}$, $b \in \mathfrak{n}^*$. By this identification the coadjoint action is realized by the formula $\text{Ad}_g^* b = P(\text{Ad}_g b)$, where P is a natural projection of the space of $n \times n$ -matrices onto \mathfrak{n}^* .

Recall that for any Lie algebra \mathfrak{g} the algebra $K[\mathfrak{g}^*]$ is a Poisson algebra with respect to the Poisson bracket such that $\{x, y\} = [x, y]$, $x, y \in \mathfrak{g}$. The symplectic leaves of this bracket coincides with the orbits of coadjoint representation [1]. Respectively, the algebra of Carimir elements of $K[\mathfrak{g}^*]$ coincides with the algebra of invariants $K[\mathfrak{g}^*]^L$ of coadjoint representation.

We also recall that the coadjoint orbits of an arbitrary nilpotent Lie group are the closed subsets with respect to the Zariski topology in \mathfrak{g}^* [5, 11.2.4].

Consider the standard basis $\{y_{ij} : n \geq i > j > 1\}$ in the algebra \mathfrak{n} . Denote by A the set of all pairs (i, j) , where $i > j$. We shall also use the notation y_ξ for y_{ij} , where $\xi = (i, j)$.

Consider the ideal \mathfrak{m} in \mathfrak{n} spanned (as a linear subspace) over the field K by some system of root vectors $\{y_{ij}, (i, j) \in M\}$, where $M \subset A$. Denote by \mathcal{L} the factor algebra $\mathfrak{n}/\mathfrak{m}$ and by L the corresponding factor group of N with respect to the normal subgroup $\exp(\mathfrak{m})$.

Note that the conjugate space \mathcal{L}^* is a subspace of \mathfrak{n}^* that consists of all $f \in \mathfrak{n}^*$ which annihilate on \mathfrak{m} . A coadjoint L -orbit for $f \in \mathcal{L}^*$ coincides with its N -orbit.

Consider the order relation \succ on the set of pairs A such that

$$(n, 1) \succ (n-1, 1) \succ \dots \succ (2, 1) \succ (n, 2) \succ \dots \succ (3, 2) \succ \dots \succ (n, n-1).$$

By the ideal \mathfrak{m} we construct the diagram which is a $n \times n$ -matrix with empty places (i, j) , $i \leq j$; the other places (i.e. places of A) are filled by symbols \otimes , \bullet , "+" and "-" following the rules stated below. The places $(i, j) \in M$ are filled by the symbol \bullet . We call this procedure the zero step of the construction of diagram.

Further, we put the symbol \otimes on the greatest place with respect to the order \succ in $A \setminus M$. Note that this symbol will be placed in the first column if the set of

the pairs of type $(i, 1)$ from $A \setminus M$ is nonempty. Assume that we put the symbol \otimes on the place (k, t) , $k > t$. Further, we put the symbol "-" on all places (k, a) , $t < a < k$, and we put the symbol "+" on all places (b, t) , $1 < b < k$. We finish the first step of constructing the diagram.

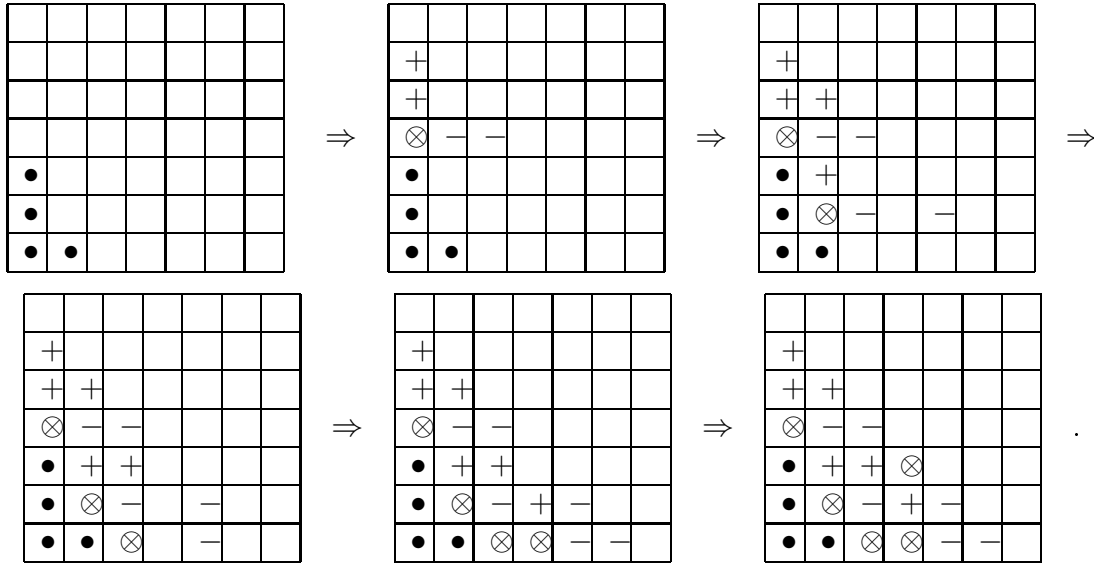
If after this procedure some places of A are not filled we again put the symbol \otimes on the greatest (with respect to the order \succ) empty place in A . Then, similarly we put the symbols "+" and "-" on the empty places taking into account the following: we put the symbols "+" and "-" in pairs; if the both places (k, a) and (a, t) , where $k > a > t$ are empty, we put "-" on the first place and "+" on the second place; if one of these places, (k, a) or (a, t) , are already filled, then we do not fill the other place. After this procedure we finish the step that we call a second step.

Continuing the procedure further we have got the diagram. We denote this diagram by $\mathcal{D}_{\mathcal{L}}$. The number of last step is equal to the number of symbols \otimes in the diagram.

Example. Let $n = 7$, $\mathfrak{m} = Ky_{51} \oplus Ky_{61} \oplus Ky_{71} \oplus Ky_{62}$. The corresponding diagram is as follows

$$\mathcal{D}_{\mathcal{L}} = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & \\ \hline & + & & & & & \\ \hline & + & + & & & & \\ \hline & \otimes & - & - & & & \\ \hline & \bullet & + & + & \otimes & & \\ \hline & \bullet & \otimes & - & + & - & \\ \hline & \bullet & \bullet & \otimes & \otimes & - & - \\ \hline \end{array}.$$

We construct the diagram in 5 steps, beginning with zero step:



Denote by S (resp. C^+ , C^-) the set of pairs (i, j) , filled in the diagram by symbol \otimes (resp. "+" "-"). The set A of pairs (i, j) , $i > j$ is decomposed into

nonintersecting subsets: $A = M \sqcup C^+ \sqcup C^- \sqcup S$. We shall give the statement of theorems 1 and 2.

Denote by \mathbb{A}_m the Poisson algebra $K[p_1, \dots, p_m; q_1, \dots, q_m]$, $\{p_i, q_i\} = 1$, and $\{p_i, q_j\} = 0$ for $i \neq j$.

Recall that a Poisson algebra \mathcal{A} decomposes into a product of two Poisson algebras $\mathcal{B}_1 \otimes \mathcal{B}_2$ if \mathcal{A} is isomorphic as a commutative associative algebra to $\mathcal{B}_1 \otimes \mathcal{B}_2$ and $\{\mathcal{B}_1, \mathcal{B}_2\} = 0$.

Theorem 1. There exist $z_1, \dots, z_s \in K[\mathcal{L}^*]^L$ where $s = |S|$ such that

- 1) every $z_i = y_{\xi_i} Q + P_{>i}$, where Q is some product of powers of z_1, \dots, z_{i-1} , and $P_{>i}$ is a polynomial in $\{y_\eta\}$, $\eta \succ \xi_i$;
- 2) denote by \mathcal{Z} the denominator subset, generated by z_1, \dots, z_s ; the localization $K[\mathcal{L}^*]_{\mathcal{Z}}$ of algebra $K[\mathcal{L}^*]$ with respect to the denominator subset \mathcal{Z} is isomorphic as a Poisson algebra to the tensor product $K[z_1^\pm, \dots, z_s^\pm] \otimes \mathbb{A}_m$ for some m .

The next section is devoted to the proof of this theorem. We shall use the procedure of step by step decomposition of Poisson algebras into tensor products $\mathbb{A}_m \otimes \mathcal{B}$. Note that in the general setting not every Poisson algebra \mathcal{A} , that contains \mathbb{A}_m , can be decomposed into a tensor product of Poisson algebras $\mathbb{A}_m \otimes \mathcal{B}$. For example, the algebra $\mathcal{A} = K[p, q, a]$ with the bracket $\{p, q\} = 1$, $\{p, a\} = a$ can't be decomposed.

Theorem 2.

- 1) The field $K(\mathcal{L}^*)^L$ of invariants coincides with the field $K(z_1, \dots, z_s)$.
- 2) Maximal dimension of the coadjoint orbits in \mathcal{L}^* is equal to the number of symbols "+" and "-" in diagonal $\mathcal{D}_{\mathcal{L}}$.
- 3) The index of Lie algebra \mathcal{L} coincides with the number of symbols \otimes in diagonal $\mathcal{D}_{\mathcal{L}}$.

Proof of the theorem 2. Theorem 1 implies that the algebra $K[z_1^\pm, \dots, z_s^\pm]$ coincides with the algebra of Cazimir elements in the Poisson algebra $K[\mathcal{L}^*]_{\mathcal{Z}}$. Hence, $K[z_1^\pm, \dots, z_s^\pm]$ is a localization of the algebra of invariants $K[\mathcal{L}^*]^L$ with respect to \mathcal{Z} . This follows 1) and further 2) and 3). \square

Proof of Theorem 1

Denote by B_i the set of pairs (a, b) , $a > b$, unfilled after the i th step in the procedure of construction of diagram. The sets B_i create the chain:

$$B_0 \supset B_1 \supset \dots \supset B_s = \emptyset,$$

where $s = |S|$. Denote $A_i = B_i \sqcup M$, $\mathbf{n}_i = \text{span}\{y_\eta : \eta \in A_i\}$, $\mathcal{L}_i = \mathbf{n}_i/\mathbf{m}$. Here $A = A_0$.

Let $S = \{\xi_1, \dots, \xi_s\}$. Recall that the place $\xi_i \in S$ is filled in the diagram by the symbol \otimes during the i th step.

For $1 \leq i \leq s$ we denote by C_i^- the set of pairs (a, b) , $a > b$, that is filled by "-" during the i th step.

Lemma 1. The subspace \mathfrak{n}_i (resp. \mathcal{L}_i) in \mathfrak{n} (resp. \mathcal{L}) is a Lie subalgebra.

Proof. We shall prove the statement using the method of induction by i . Suppose the the statement is true for all numbers less than i and the statement is false for number i . Then there exists $n \geq a > b > c \geq 1$ such that $(a, b) \in B_i$, $(b, c) \in B_i$, and

$$(a, c) \in \bigsqcup_{k \leq i} C_k^- \sqcup \bigsqcup_{k \leq i} C_k^+ \sqcup \{\xi_1 \dots \xi_i\}.$$

Case $(a, c) \in \{\xi_1 \dots \xi_i\}$ is not possible, because if it occurs then the places (a, b) and (b, c) are filled by the symbols "-" and "+" respectively; so the both pairs will not lie in B_i . Consider two last cases.

Case 1. $(a, c) \in C_k^-$ for some $k \leq i$. Then $\xi_k = (a, t)$ for some $t < c$. During the k th step the place (c, t) is filled by the symbol "+" and the place (a, c) – by the symbol "-".

After the $(k-1)$ th step the place (b, t) is filled, otherwise during the k th step we put the symbol "+" on the place (b, t) and the symbol "-" on the place (a, b) and obtain $(a, b) \notin B_i$.

So, after the $(k-1)$ th step the places (c, t) and (b, c) are empty, and the place (b, t) is filled by one of the symbols \otimes , "+" or "-". This contradicts to the assumption that \mathfrak{n}_{k-1} is a subalgebra.

Case 2. $(a, c) \in C_k^+$ for some $k \leq i$. This case is considered similarly. \square

Denote by \mathfrak{d}_i^- , where $1 \leq i \leq s$, the linear subspace in \mathfrak{n} , spanned by the vectors y_η such that $\xi_i \succ \eta$ и

$$\eta \in \bigsqcup_{1 \leq j \leq i} C_j^-.$$

Remark. In the above example:

$$\xi_1 = (4, 1), \quad \xi_2 = (6, 2), \quad \xi_3 = (7, 3), \quad \xi_4 = (7, 4), \quad \xi_5 = (5, 4);$$

$$\begin{aligned} C_1^- &= \{(4, 2), (4, 3)\}, & \mathfrak{d}_1^- &= \text{span}\{y_{42}, y_{43}\}; \\ C_2^- &= \{(6, 3), (6, 5)\}, & \mathfrak{d}_2^- &= \text{span}\{y_{42}, y_{43}, y_{63}, y_{65}\}; \\ C_3^- &= \{(7, 5)\}, & \mathfrak{d}_3^- &= \text{span}\{y_{43}, y_{63}, y_{65}, y_{75}\}; \\ C_4^- &= \{(7, 6)\}, & \mathfrak{d}_4^- &= \text{span}\{y_{65}, y_{75}, y_{76}\}; \\ C_5^- &= \emptyset, & \mathfrak{d}_5^- &= \mathfrak{d}_4^-. \end{aligned}$$

Lemma 2. For every $\xi \in S$ the linear subspace \mathfrak{d}_ξ^- is a subalgebra in \mathfrak{n} .

Proof. Suppose that the statement is true for all $l < i$. We shall prove it for the

number i . The proof follows from the points 1) and 2) below.

1) Let $(k, b) \in C_i^-$ and $(a, k) \in C_l^-$ for some $l < i$. We shall show that (a, b) lies in C_m^- for some $m < i$.

Assume the contrary. Then $(a, b) \in B_i$ (i.e. the place (a, b) is empty after the i th step). Since $(k, b) \in C_i^-$, then $\xi_i = (k, t)$ for some $t < b$ and the pair $(b, t) \in C_i^+ \subset B_{i-1}$. So, the pairs (a, b) and (b, t) lie in B_{i-1} . Since \mathbf{n}_i is a subalgebra, then $(a, t) \in M \sqcup B_{i-1}$.

The pair (a, t) can't lie in M since in this case all (a, j) , $j \leq t$, lie in M . This contradicts to an existence of $l < i$ such that $(a, k) \in C_l^-$.

If $(a, t) \in B_{i-1}$, then $(a, t) \succ (k, t)$ (since $a > k$). We obtain a contradiction to the choice of $\xi_i = (k, t)$ as a greatest pair in the sense of order relations \succ in B_{i-1} .

2) Let $(k, a) \in C_i^-$ and $(a, b) \in C_l^-$ for some $l < i$. Let us show that (k, b) lies in C_m^- for some $m < i$.

Assume the contrary. Then $(k, b) \in B_i$ (i.e. the place (k, b) is empty after the i th step). As above $\xi_i = (k, t)$ for some $t < b$. The pair $(b, t) \notin B_{i-1}$. Otherwise during the i th step the place (k, b) will be filled by the symbol "-" and (b, t) - by the symbol "+".

So, the place (b, t) is filled before the i th step. According to the procedure of placing the symbols, the symbol \otimes can't take the place (b, t) . The symbol "-" also can't take the place (b, t) , since in this case $(b, t) \in C_m^-$ for some $m < i$ and we have got the pair of places $(a, b) \in C_l^-$ and $(b, t) \in C_m^-$, where $l, m < i$, such that the place (a, t) is empty after the $(i-1)$ th step (during the i th step we put the symbol "+" on the place (a, t)). This contradicts to the induction conjecture.

It remains to consider the last case when the place (b, t) is filled by the symbol "+". Then there exists a place (p, t) , $p > k$, that is filled by the symbol \otimes during some step $q < i$; at the same time the place (p, b) is filled by the symbol "-" and the places (p, a) and (p, k) are filled by "-" before the q th step. Finally, after the $(q-1)$ th step we obtain the pair of places (p, a) and (a, b) filled by the symbol "-" and the empty place (p, b) (it is filled by the symbol "-" during the q th step). A contradiction. \square

Proof of the theorem 1. Consider the localization $S(\mathcal{L}_{i-1})_z$ of the algebra $S(\mathcal{L}_{i-1})$ with respect to the denominator subset, generated by $z = y_{\xi_i}$. Let us show that for any i there exists an embedding of Poisson algebras

$$\theta_{i-1} : S(\mathcal{L}_i) \rightarrow S(\mathcal{L}_{i-1})_z \quad (1)$$

such that the Poisson algebra $S(\mathcal{L}_{i-1})_z$ decomposes

$$S(\mathcal{L}_{i-1})_z = \mathbb{A}_m \otimes K[z^{\pm}] \otimes \theta_{i-1} S(\mathcal{L}_i), \quad (2)$$

for some m . Then applying the induction by i we finish the proof of 2). The proof of 1) follows from the construction of the embedding θ_{i-1} that we shall present below.

Let $\xi_i = (k, t)$, $k > t$. There exists $p > k$ such that the pairs of the t th column from M fill $\{(j, t) : p \leq j \leq n\}$.

Decompose the set B_i into subsets

$$B_i = B_i^{(1.1)} \sqcup B_i^{(1.2)} \sqcup B_i^{(2)} \sqcup B_i^{(3)} \sqcup B_i^{(4)}, \quad (3)$$

Here every subset is contained of the pairs $(a, b) \in B_i$ that satisfy the additional conditions:

1.1) $(a, b) \in B_i^{(1.1)}$, if $1 < a < k$ and $t < b < k$, and both pairs (a, t) and (k, b) lie in B_{i-1} ;

1.2) $(a, b) \in B_i^{(1.2)}$, if one these conditions take place:

a) $1 < a < k$ and $b = t$,

b) $a = k$ and $t < b < k$,

c) $1 < a < k$ and $t < b < k$, at that $(a, t) \notin B_{i-1}$ or $(k, b) \notin B_{i-1}$;

2) $(a, b) \in B_i^{(2)}$, if $k < a \leq n$, $t < b < k$;

3) $(a, b) \in B_i^{(3)}$, if $p \leq a \leq n$, $b = k$;

4) $(a, b) \in B_i^{(4)}$, if $k < a \leq n$, $b > k$.

Remark. Note that the pairs (a, k) , $k < a < p$ do not lie in B_{i-1} (precisely, they are filled by the symbol "-" during the steps with numbers $\leq i-1$). Note also that all (a, j) , where $p \leq a \leq n$ and $t < j \leq k$, lie in $A_{i-1} = B_{i-1} \sqcup M$.

We introduce the following notations:

$$\tilde{y}_{ab} = - \begin{vmatrix} y_{at} & y_{ab} \\ y_{kt} & y_{kb} \end{vmatrix} \cdot y_{kt}^{-1} \quad \text{for } (a, b) \in B_i^{(1.1)}; \quad (4)$$

$$\tilde{y}_{ak} = \left(y_{ak}y_{kt} + \sum_{(j,t) \in B_{i-1}} y_{aj}y_{jt} \right) y_{kt}^{-1} \quad \text{for } (a, k) \in B_i^{(3)}; \quad (5)$$

$$\tilde{y}_{ab} = y_{ab} \quad \text{для всех остальных } (a, b) \in B_i. \quad (6)$$

We extent the correspondence

$$\theta_{i-1} : y_{ab} \mapsto \tilde{y}_{ab}, \quad (a, b) \in B_i$$

to the embedding (1). By definition, the subalgebra \mathbb{A}_m is generated by the elements $p_j = y_{kj}$, $(k, j) \in C_i^-$, and $q_j = y_{jt}y_{kt}^{-1}$, $(j, t) \in C_i^+$.

Easy to see that $S(\mathcal{L}_{i-1})_z$ decomposes as a commutative associative algebra into the tensor product $\mathbb{A}_m \otimes K[z^\pm] \otimes \theta_{i-1}S(\mathcal{L}_i)$. The subalgebras \mathbb{A}_m , $K[z^\pm]$ and

$\theta_{i-1}S(\mathcal{L}_i)$ are pairwise in involution (recall that two elements u and v of a Poisson algebra are in involution if $\{u, v\} = 0$).

It remains to show that θ_{i-1} is an embedding of Poisson algebras. It is necessary and sufficient to check that

$$\{\tilde{y}_{ad}, \tilde{y}_{bc}\} = \delta_{db}\tilde{y}_{ac} \bmod \mathfrak{m}, \quad (7)$$

for all $a > d, b > c, a > c$ where (a, d) and (b, c) lie in B_i .

We prove the condition (7) by running over all case of entering of pairs (a, d) , (b, c) in the subsets of decomposition (3).

Case 1. $(a, d) \in B_i^{(4)}$. One can prove (7) easily.

Case 2. $(a, d) \in B_i^{(3)}, (b, c) \in B_i^{(2)}$.

In this case, $d = k, a \geq p$ and $\{y_{ak}, y_{bc}\} = 0$. From (6) we obtain $\tilde{y}_{bc} = y_{bc}$. We have to show that

$$\{\tilde{y}_{ak}, y_{bc}\} = 0 \bmod \mathfrak{m}. \quad (8)$$

The element y_{bc} is in involution with all elements of the standard basis from (5) apart from y_{ct} (if this element really appears in (5)). If $b \geq p$, then $\{y_{bc}, y_{ct}\} = y_{bt} \in \mathfrak{m}$, this proves (8).

Let $k < b < p$. Let us show that an existence of $(b, c) \in B_i$, where $k < b < p$, implies that $(c, t) \notin B_{i-1}$. Really, let $(c, t) \in B_{i-1}$. The pair (b, c) lies in B_i and, therefore, also it lies in B_{i-1} . Since \mathfrak{n}_i is a subalgebra, then $(b, t) \in B_{i-1}$. On the other side, all places $(j, t), k < j < p$, are already filled after $(i-1)$ th step. That is no one of these pairs are contained in B_{i-1} (see the above remark). A contradiction. So $(c, t) \notin B_{i-1}$ and, therefore, y_{ct} does not appear in (5). The equality (8) is true.

Case 3. $(a, d) \in B_i^{(3)}, (b, c) \in B_i^{(1.1)}$.

As above $d = k, a \geq p$ and $\{y_{ak}, y_{bc}\} = 0$. We have to prove that

$$\{\tilde{y}_{ak}, \tilde{y}_{bc}\} = 0 \bmod \mathfrak{m}, \quad (9)$$

where \tilde{y}_{ak} and \tilde{y}_{bc} from (5) and (4) respectively.

The element y_{bc} is in involution with all elements of the standard basis from (5) apart from y_{ab} and y_{ct} (if these elements really appear in (5)). Since $(b, c) \in B_i^{(1.1)}$, then $y_{bt} \in B_{i-1}$ and the element y_{ab} really appears in the sum of (5).

As for the element y_{ct} we can't say this exactly. If $(c, t) \in B_{i-1}$, then the element y_{ct} appears in the sum of (5). The equality (9) is fulfilled since all elements of the standard basis from (4) are in involution with \tilde{y}_{ak} .

If $(c, t) \notin B_{i-1}$, then the element y_{ct} does not appear in the sum of (5). Since

$a \geq p$, then $y_{at} \in \mathfrak{m}$; the equality (9) is checked directly:

$$\{\tilde{y}_{ak}, \tilde{y}_{bc}\} = -\{y_{ak}y_{kt} + y_{ab}y_{bt}, \begin{vmatrix} y_{bt} & y_{bc} \\ y_{kt} & y_{kc} \end{vmatrix}\} \cdot y_{kt}^{-2} = 0 \bmod \mathfrak{m}.$$

Case 4. $(a, d) \in B_i^{(3)}$, $(b, c) \in B_i^{(1,2)}$.

As above $d = k$. Recall that $B_i^{(1,2)}$ decomposes into three subsets (see. a), b), c) from the definition of $B_i^{(1,2)}$).

4a. $b = k$ and $t < c < k$. Then $\{y_{ak}, y_{kc}\} = y_{ac}$. The element y_{kc} are in involution with all elements of the standard basis from (5) apart from y_{ak} and y_{ct} (if the last element really appears in (5)). In our case the place (c, t) is filled after the $(i-1)$ th step (otherwise during the i th step the place (c, t) will obtain the symbol "+" and the place (k, c) – the symbol "-"). The element y_{ct} does not appear in (5). Hence $\{\tilde{y}_{ak}, y_{kc}\} = 0$.

4b. $c = t$ and $1 < b < k$. Then $\{y_{ak}, y_{bt}\} = 0$. The element y_{bt} is in involution with all elements of the standard basis from (5) apart from y_{ab} . Since $y_{at} \in \mathfrak{m}$, then

$$\{\tilde{y}_{ak}, y_{bt}\} = -\{y_{ab}y_{bt}, y_{bt}\}y_{kt}^{-1} = -y_{at}y_{bt}y_{kt}^{-1} = 0 \bmod \mathfrak{m}.$$

4c. $1 < b < k$ and $t < c < k$, at that $(b, t) \notin B_{i-1}$ or $(k, c) \notin B_{i-1}$. Since in this case $\{y_{ak}, y_{bc}\} = 0$ and $\tilde{y}_{bc} = y_{bc}$, then we have to show that

$$\{\tilde{y}_{ak}, y_{bc}\} = 0. \tag{10}$$

If (b, t) and (c, t) lie in B_{i-1} , then the calculation

$$\{\tilde{y}_{ak}, y_{bc}\} = -\{y_{ab}y_{bt} + y_{ac}y_{ct}, y_{bc}\}y_{kt}^{-1} = 0$$

proves (10).

If the both pairs (b, t) and (c, t) do not lie in B_{i-1} , then y_{bc} is in involution with all elements of the standard basis from (5). This proves (10).

The case $(b, t) \notin B_{i-1}$, $(c, t) \in B_{i-1}$ is not possible, since in this case the both pairs (b, c) and (c, t) lie B_{i-1} , but (b, t) do not. This contradicts to the fact that \mathfrak{n}_{i-1} is a subalgebra.

Let us show that the case $(b, t) \in B_{i-1}$, $(c, t) \notin B_{i-1}$ is also not possible. Really, $(b, t) \in B_{i-1}$ and $(b, c) \in B_i^{(1,2)}$ imply $(k, c) \notin B_{i-1}$. That is the place (k, c) is filled after the $(i-1)$ th step (by the symbol "-"). Since $(c, t) \notin B_{i-1}$, then the place (c, t) is also filled after the $(i-1)$ th step. The symbol \otimes can't take this place because of the procedure of placing the symbols. The symbol "-" also can't take this place because in this case after the $(i-1)$ th step we have got symbol "-" on the places (k, c) and (c, t) , and the place (k, t) is empty (during the i th step this place is filled by \otimes). This contradicts to lemma 2.

It remains the last case: the place (c, t) is filled by the symbol "+". There exists a pair $\xi_l = (q, t) \in S$, $q > k$, $l < i$. According to the procedure of placing of symbols the place (q, k) is filled by the symbol "-" before the l th step (otherwise the place (k, t) is filled by the symbol "+" and the place (q, k) , respectively, by "-"). The symbol "-" appears on the place (k, c) also before the l th step. We have got that after the $(l - 1)$ th step the symbol "-" on the places (q, k) and (k, c) , but the place (q, t) is empty. This contradicts to lemma 2 for $i = l - 1$.

The equality (7) is obvious for cases $(a, d) \in B_i^{(2)}$, $(b, c) \in B_i^{(2)}$. Denote $B_i^{(1)} = B_i^{(1.1)} \sqcup B_i^{(1.2)}$.

Case 5. $(a, d) \in B_i^{(2)}$, $(b, c) \in B_i^{(1)}$.

Recall that here $\tilde{y}_{ad} = y_{ad}$. The equality (7) is obvious for the case $(b, c) \in B_i^{(1.2)}$. Consider the case $(b, c) \in B_i^{(1.1)}$. One can easily check (7) for the case $d \neq b$ and for the case $a > p$.

Let us prove that the last case $d = b$, $k < a < p$, $(a, b) \in B_i^{(2)}$, $(b, c) \in B_i^{(1.1)}$ is not possible.

From the above remark we see that all pairs of the form (j, k) , $k < j < p$, do not lie in B_{i-1} . Precisely, all (j, k) , $k < j < p$, are filled by the symbol "-" before the $(i - 1)$ th step. This concerns the pair (a, k) .

On the other hand, since $(b, c) \in B_i^{(1.1)}$, then after the i th step the place (b, t) is either empty, or is filled by the symbol "+". In any case after the i th step the place (k, b) is filled by the symbol "-".

Finally, after the i th step the places (a, k) and (k, b) are filled by the symbol "-". Then by lemma 2 the place (a, b) must be filled by the symbol "-" during the $(i - 1)$ th step. This contradicts to $(a, b) \in B_i$.

Case 6. $(a, d) \in B_i^{(1)}$, $(b, c) \in B_i^{(1)}$.

The equality (7) is obvious when two pairs lie in $B_i^{(1.2)}$ and can be easily checkable in the case when two pairs lie in $B_i^{(1.1)}$. One can also easily check (7) in the case $d \neq b$ and in the case when one of the pairs lies in $B_i^{(1.2)}$ and satisfies the points a) and b) (see the definition of $B_i^{(1.2)}$). Consider the last cases.

i) Let $(a, b) \in B_i^{(1.1)}$ and $(b, c) \in B_i^{(1.2)}$ where $(k, c) \notin B_{i-1}$ or $(b, t) \notin B_{i-1}$ (see condition 1.2(c)). Let us show that in this case $(a, c) \in B_i^{(1.1)}$. Then (7) is easily checkable.

Suppose the contrary $(a, c) \in B_i^{(1.2)}$. Then one of the places (a, t) or (k, c) is filled after the $((i - 1))$ th step. Since $(a, b) \in B_i^{(1.1)}$, then the place (a, t) is empty. Hence the place (k, c) is filled after the $(i - 1)$ th step. On the other hand, $(a, b) \in B_i^{(1.1)}$ implies that (k, b) is empty after the $(i - 1)$ th step. The place (b, c)

is also empty. Since \mathfrak{n}_{i-1} is a subalgebra, then the place (k, c) is empty after the $(i - 1)$ th step. A contradiction.

ii) $(a, b) \in B_i^{(1.2)}$ and $(b, c) \in B_i^{(1.1)}$, where $(k, b) \notin B_{i-1}$ or $(a, t) \notin B_{i-1}$ (see condition 1.2(c)). Let us show that $(a, c) \in B_i^{(1.1)}$ (this implies (7)).

Really, $(b, c) \in B_i^{(1.1)}$ implies $(k, c) \in B_{i-1}$. The place (a, t) can't be filled after the $(i - 1)$ th step because at the same time the places (a, b) and (b, t) are empty. Since the pairs (a, t) and (k, c) lie in B_{i-1} we have got $(a, c) \in B_i^{(1.1)}$. \square

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